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# HYPERBOLIC POLYGONAL SPIRALS

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## HYPERBOLIC POLYGONAL SPIRALS

Jillian Russo

**Abstract.** This article is based on the construction of Nested Hyperbolic Polygonal Spirals. The construction uses constructible Euclidean angles to create hyperbolic polygons of five or more sides. The nested polygons are formed by connecting the midpoints of the sides of the original polygon, thus creating a spiral. The construction is included for the readers to be able to construct one for themselves as they read along. This construction, along with hyperbolic trigonometric formulas, led to the results: measures of the angles, side lengths and areas of all the parts of the spiral. Furthermore, the construction is used to prove the constructible hyperbolic regular polygons have the same number of sides as the constructible Euclidean polygons.

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## 1 Introduction

In this paper we construct and examine what we have named *nested hyperbolic polygonal spirals* within the Poincaré Disk (Figure 1). The construction is easy to follow, so the readers will be able to construct one for themselves as they read along. My advisor and I studied the areas, angles and side lengths of the parts of the spiral. Also, the construction leads to a proof describing which regular hyperbolic polygons with all right angles are constructible.

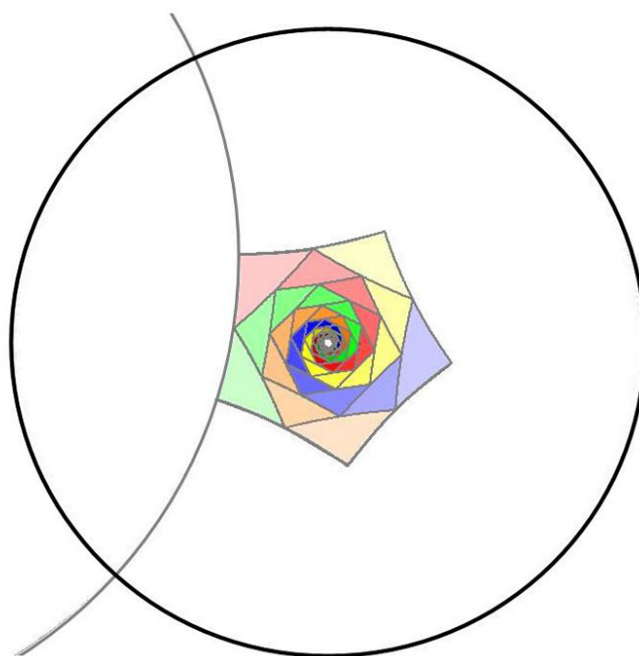


Figure 1. Nested hyperbolic pentagons.

Hyperbolic geometry dates back to the nineteenth century with Bolyai and Lobachevsky being credited as the cofounders. This geometry comes from a change in the parallel postulate. Instead of the Euclidean requirement that for every given line and a given point not on that line, there is one parallel line through the given point, in hyperbolic geometry there are many parallel lines through a given point (P) not on a given line ( $l$ ) (Figure 2.) There are different models for representing this type of geometry, but for our purposes we focused on the Poincaré Disk Model.

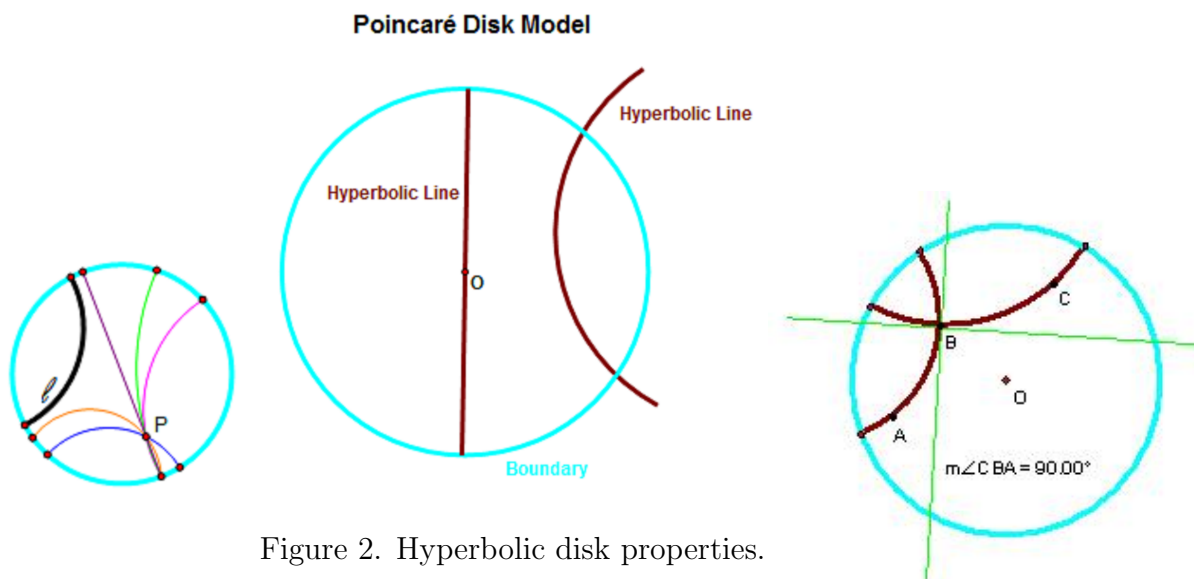


Figure 2. Hyperbolic disk properties.

In the disk model, the hyperbolic space is confined by a boundary circle. There are two kinds of hyperbolic lines: diameters and arcs of circles orthogonal to the boundary. In this space, hyperbolic lines may appear bent in the Euclidean sense, but in hyperbolic space they are straight lines. Also, angles within this geometry are measured using their tangents. This creates the characteristic property that the sum of the angles of a hyperbolic triangle is between 0 and  $\pi$ . Furthermore, all similar triangles are congruent and non-intersecting lines have exactly one mutual perpendicular line.

We can now examine Figure 1 with the vocabulary of hyperbolic geometry. The complete circle is the boundary of the space. The outermost pentagon is the level 1 hyperbolic pentagon. It has all right angles. One side has been drawn with the hyperbolic line containing it. The other lines have been hidden in order to keep the drawing simple. Any nested pentagon has the midpoints of the sides of the previous pentagon for vertices, giving us the  $k$  levels we use throughout the paper. The nested pentagons form hyperbolic triangles; we will find properties of one triangle from each level. The shading is to emphasize choosing one triangle from each level.

Here are a few more hyperbolic geometry facts we will use without proof.

- We can have a hyperbolic polygon with all right angles and as many sides as we want, as long as we want more than four.
- The nested polygons have a limiting case as they are constructed inward: the Euclidean regular polygon of that many sides. The hyperbolic segments get Euclidean straighter as the polygons get smaller.

- As the polygons go inward, the hyperbolic length of a side approaches zero. The vertex angles grow from  $\frac{\pi}{2}$  to the Euclidean limit.

The nested polygons are not congruent and do not form a tiling. The initial vertex angles do not have to be  $\frac{\pi}{2}$ ; they could be any size less than the Euclidean value of  $\frac{n-2}{n}\pi$ , where  $n \geq 5$ .

In the second section, we recall a few useful hyperbolic trigonometric identities. We give instructions for the construction of this figure in the third section. Our fourth section has area calculations, including a telescopic series. The proof that the sum of the hyperbolic lengths of one side of each level polygon converges takes up the fifth section. We close with our constructibility proof.

## 2 Hyperbolic Trigonometry

The hyperbolic disk has its own trigonometry, using  $\sinh x$  and  $\cosh x$ . We only need two hyperbolic identities, available in any hyperbolic trigonometry text [Kennelly]. We use triangle  $ABC$  whose sides are hyperbolic segments with hyperbolic lengths  $a, b, c$ .

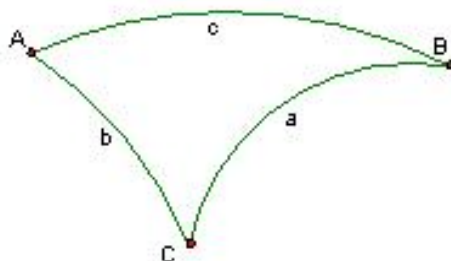


Figure 3. Hyperbolic triangle  $ABC$ .

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C \quad (1)$$

The equation above resembles the Law of Cosines in that  $C$ , the only angle involved, corresponds with the isolated side on the left hand of the equation. Since the formula uses  $\cosh a$  instead of the hyperbolic length  $a$ , we will use  $\cosh a$  in our calculations much more than  $a$ , even though we will investigate the hyperbolic lengths of the sides of our nested polygons.

We will also use the special structure of our polygons to simplify the second useful identity.

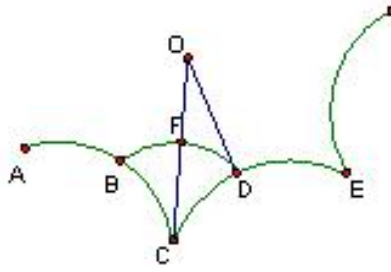


Figure 4. Level  $k$  sketch of  $\triangle OFD$ .

In Figure 4, the point  $O$  is the center of our hyperbolic polygons and our disk. Euclidean arcs  $AC$  and  $CE$  represent two consecutive sides of a level  $k$  hyperbolic polygon and arc  $BD$  represents a side of the hyperbolic polygon at the next level,  $k + 1$ . Using the notation and relationships in this figure, we have the following equation.

$$\sin \angle OFD \sin \angle ODF \cosh FD = \cos \angle DOF + \cos \angle OFD \cos \angle ODF \quad (2)$$

The triangle  $OFD$  has specific values so that we can simplify quickly. We will take segment  $\overline{BD}$  as length  $L_k$ , and the central angle connecting consecutive midpoints of a level  $k$  polygon as  $A_k$ , making  $\angle ODF = \frac{A_k}{2}$ . Since  $\angle OFD = \frac{\pi}{2}$ , formula (2) becomes

$$\sin \frac{A_k}{2} \cosh \frac{L_k}{2} = \cos \frac{\pi}{n}. \quad (3)$$

With these formulas, we calculated angles, areas, and lengths for as many levels as we wanted. We used  $\cosh L_k = S_k$  for all our calculations and notes and converted to hyperbolic length only when we wanted actual hyperbolic lengths. Our empirical investigation pointed the way to the theoretical results in this paper. The level one side length,  $S_1$ , of the pentagon is worth looking at.

**Lemma 1** *For the five-sided, all right-angled, regular hyperbolic pentagon,*

$$\cosh L_1 = S_1 = \varphi \quad (\text{the Golden Ratio.})$$

**Proof.** *From equation (2) applied to the hyperbolic triangle  $OGE$  in Figure 8 (below), we have  $\cosh GE \sin \frac{\pi}{4} \sin \frac{\pi}{4} = \cos \frac{2\pi}{5} + \cos \frac{\pi}{4} \cos \frac{\pi}{4}$ . This implies  $S_1 = \cosh GE = 2 \cos \frac{2\pi}{5} + 1$ . A trig ratio from regular Euclidean pentagons gives us  $\cos \frac{\pi}{5} = \frac{\varphi}{2}$ . A double angle formula finishes our proof:  $2 \cos \frac{2\pi}{5} + 1 = \varphi^2 - 2 + 1 = \varphi$ . ■*

### 3 Nested Hyperbolic Polygons

We will now provide an algorithm to construct a regular, right angled hyperbolic polygon and create layers of polygons inside it using the midpoints of the sides. The construction is

simple (provided the initial angle is constructible) and can be done with a series of Euclidean moves using a compass and straight edge. This limitation of the first angle forms a crucial part of our work in the last section. Figures 5, 6, and 7 below will illustrate our procedure in the  $n = 5$  case.

### Regular Right Angled Hyperbolic Polygon Construction

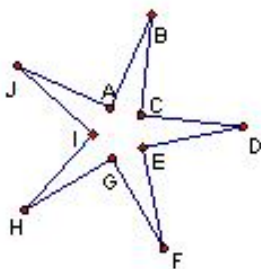


Figure 5.

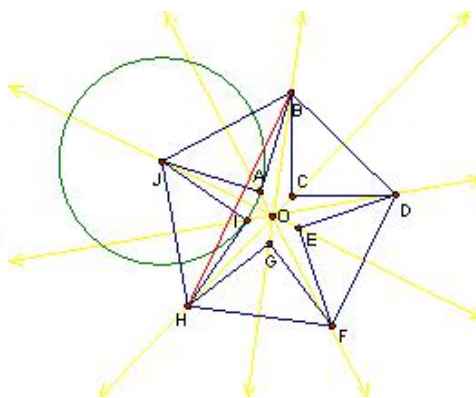


Figure 6. Boundary location.

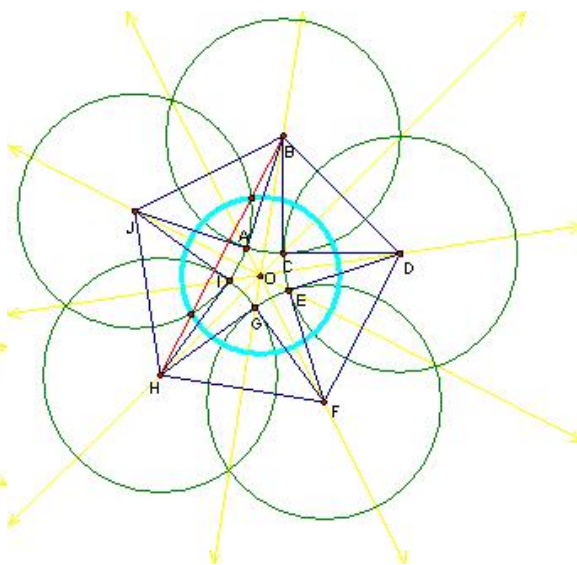


Figure 7. Polygon construction.

- Before drawing anything, choose the number of sides and calculate  $(\frac{180(n-2)}{n})$ .
- Create a segment  $AB$  of any length  $R$ . Rotate the line by the angle  $(\frac{180(n-2)}{n}) - 90$  around point  $B$  with length  $R$ , creating segment  $BC$ . Thus begins Figure 5.
- Now rotate the segment  $BC$  a negative 90 degrees around point  $C$ , creating segment  $CD$ .
- We continue the first two steps repeatedly, rotating around the newest created endpoint, until the sides close in on each other, creating Figure 5.
- Connect the vertices at the angles of measure  $(\frac{180(n-2)}{n}) - 90$ ; it will create an  $n$ -sided polygon with interior angles  $(\frac{180(n-2)}{n})$ . As we see in Figure 6, we have created  $n$  isosceles right triangles, each with hypotenuse an outer side, like  $BD$ . The rotation  $(\frac{180(n-2)}{n}) - 90$  is just right to get such a figure for  $n$  sides.
- Connect two vertices of the  $n$ -sided polygon, skipping one in between, like the segment  $BH$ .
- Now we can construct the first side of the hyperbolic regular polygon. Use a vertex,  $J$ , of the  $n$ -sided polygon as the center. Use radius of length  $R$ , so the hyperbolic line will go through the two points of length  $R$  away from the chosen vertex.
- Now construct the diagonals of the  $n$ -sided polygon to find the center. (Lines through  $O$  in Figures 6 and 7)



- The center of the hyperbolic space will be the center of the  $n$ -sided polygon ( $O$ ), where all the diagonals meet. To find the radius of the disk, we follow the segment  $BH$  we connected between two vertices (skipping one), starting at one of the vertices,  $B$ , and traveling toward the other until we hit the hyperbolic line. This unlabeled intersection gives us the disk radius distance.

- Finish the polygon by creating the hyperbolic lines as before. Use the vertices of the  $n$ -sided polygon with radii of length  $R$ . (Figure 7)

- To make the first nested polygon, we start by creating lines perpendicular to the diagonals at all the vertices of the  $n$ -sided polygon. The intersections of these perpendicular lines create a new  $n$ -sided polygon. (Figure 8 - Pentagon MNPRS)

Note: The midpoints of the arcs of the hyperbolic lines are the points where the diagonals intersect the arcs.

- Use the vertices of the newly created  $n$ -sided polygon as the center of the circles for the sides of the nested hyperbolic polygon. The radius will be the length from the vertices to the midpoint of the arc of one of the circles containing that point. (It will go through both midpoints).

- Repeat the last step for all vertices to finish the first nested hyperbolic polygon.

- Use the same three steps to construct multiple nested hyperbolic polygons always building off the previous step.

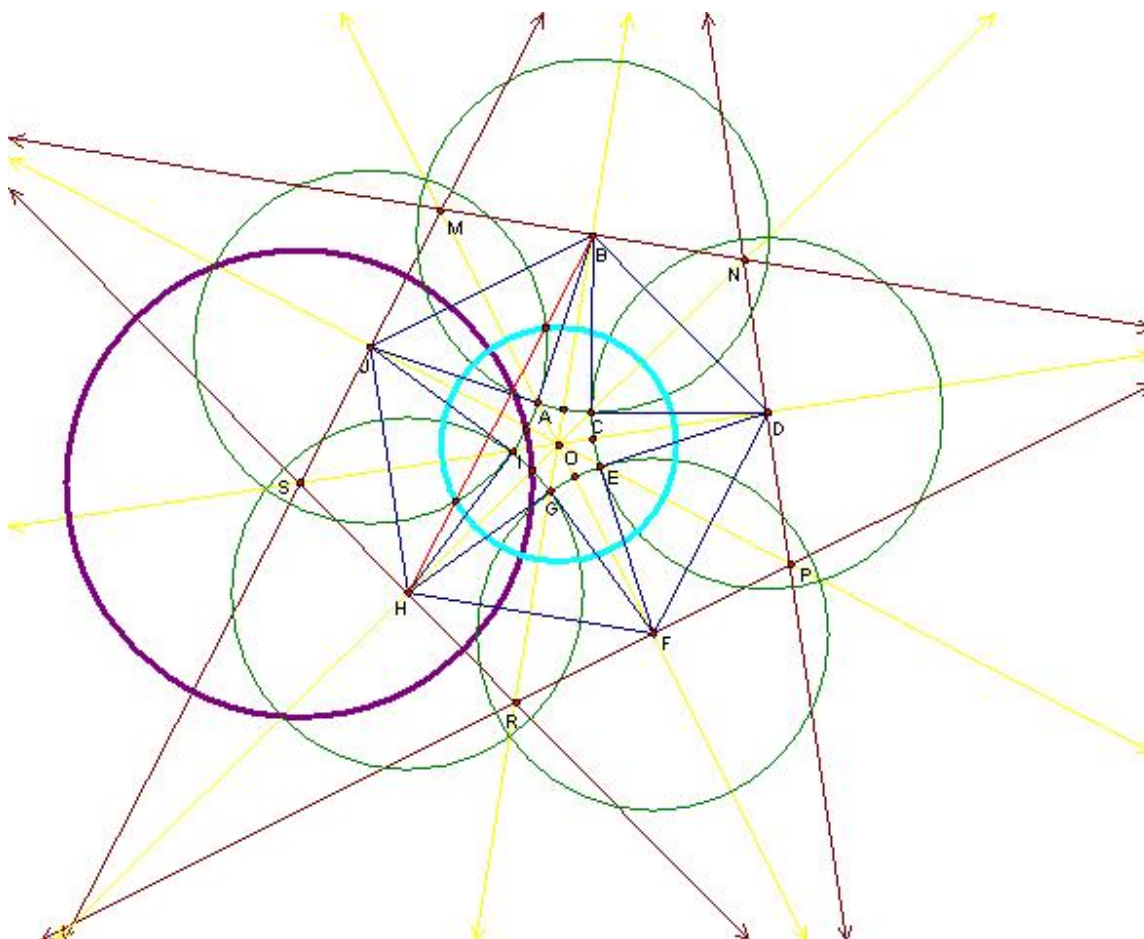


Figure 8. Nested polygon construction.

These Euclidean steps make hyperbolic objects. These steps may be used for any number of sides, given that first angle. We will now justify the key steps in the construction.

**Lemma 2**      *The level 1 construction gives a chain of mutually orthogonal circles.*

**Proof.**      *We constructed neighboring circles to have perpendicular radii. For example,  $\angle JAB = \frac{\pi}{2}$ . ■*

**Lemma 3**      *The level 1 construction gives hyperbolic lines.*

**Proof.**      *The circle with center  $J$  has two (unlabeled) points on Euclidean segment  $\overline{HB}$ . Since the circle  $J$  is orthogonal to circle  $B$ , these two unlabeled points are a point and its inverse. Any circle through a point and its inverse is orthogonal to the circle of inversion. The circle with center  $O$  passes through these two points. Therefore circles  $J$  and  $B$  contain hyperbolic lines in the disk bounded by circle  $O$ . ■*

**Lemma 4**      *The level  $k$  construction gives hyperbolic lines.*

**Proof.**      *The midpoints at any level lie on  $n$  alternating Euclidean rays (of the  $2n$  rays we construct) through  $O$ . Two consecutive midpoints determine one hyperbolic line. The Euclidean ray from  $O$  through one midpoint passes through the circle containing that midpoint again, at the inverse of that point. The midpoint, the center of the containing circle and its inverse are Euclidean collinear with  $O$ . By constructing the center of the next level using the intersection of the Euclidean perpendicular lines at vertices of the previous level Euclidean polygon, we get the next level circle through the midpoint and its inverse, making the level  $k$  circle contain a hyperbolic line. (In our Figure 8, circle  $S$  passes through a midpoint and its inverse.) ■*

We are relying on a property of orthogonal circles which can be found in [Goodman-Strauss], that a circle through a point and its inverse must be orthogonal to the circle of inversion. We depend on the diagonals which skip one vertex, like  $\overline{HB}$ , to find the centers of the circles which contain our hyperbolic lines. The Euclidean segment  $\overline{HB}$  passes through the intersection of two circles and is called the *radical axis* of circles  $J$  and  $O$ .

**Lemma 5**      *When one circle is a hyperbolic line and the other is the boundary, then the radical axis contains the center of any hyperbolic line orthogonal to the given hyperbolic line (except the diameter, of course.)*

**Proof.**      *Without loss of generality, consider the circle centered at  $J$  in Figure 8. We have already built circle  $J$  orthogonal to the boundary. We can change our point of view and see that we have circle  $O$  orthogonal to circle  $J$ , treating circle  $J$  as the boundary, temporarily, of a hyperbolic disk. The two unlabeled points form a point and its inverse across circle  $J$ . ■*

This use of the radical axis is also given as a homework problem on the inversion of circles in [Venema]. We want to emphasize that this construction works for any number of sides, given that first Euclidean angle.

## 4 The Area of One Spiral

Figure 1 shows five spirals made from hyperbolic triangles, one from each level, distinguished by color. The sum of the areas of these triangles is easily calculated. First, we need the area formula for a hyperbolic triangle  $ABC$ . With angles in radians, the area is  $\pi - (A + B + C)$ . (There is a multiplication constant which we will take as 1.)

In Figure 1, we see 5 spirals all converging at  $O$ . The picture would look similar for  $n$  sides, except the level 1 polygon gets bigger as  $n$  increases (see Section 5 for the limit.) Then the spirals fill the interior of the first level polygon, with no overlap. For  $n$  sides, the spiral consists of  $\frac{1}{n}$  of the area of the first level polygon. The hyperbolic area formula of the first level is

$$(n - 2) \pi - \frac{n\pi}{2} = \frac{n\pi}{2} - 2\pi.$$

We can divide by  $n$  to see the spiral has area  $\frac{\pi}{2} - \frac{2\pi}{n}$ .

A more interesting calculation can be made for this area, this time applying the hyperbolic area formula at each level and adding. We will get an honest-to-goodness telescopic series whose sum is, again,  $\frac{\pi}{2} - \frac{2\pi}{n}$ . A telescopic sum has terms which cancel throughout the series, usually leaving a front term and a back term, as we will see in our example. We begin with a figure of two consecutive levels in a polygon of  $n$  sides.

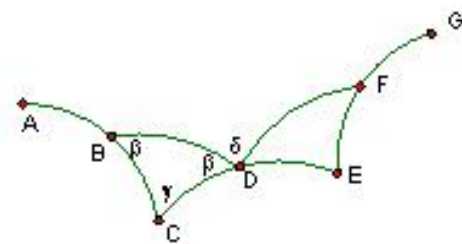


Figure 9. Supplementary angles at  $D$ .

Using the area formula for a hyperbolic triangle again, we find the area of triangle  $BCD$  is

$$\pi - (\gamma + 2\beta) = \pi - (\gamma + 2(\pi - \delta) \div 2) = \delta - \gamma. \quad (4)$$

Now, these two angles  $\gamma$  and  $\delta$  are just the vertex angles of isosceles hyperbolic triangles from two consecutive levels. The sum of the areas of hyperbolic triangles in the spiral would be a telescopic sum where the surviving terms would be the limit of the innermost angle minus the outermost angle. Luckily, both of these are available. The nested polygons approach a Euclidean regular polygon of  $n$  sides. The outermost angle is a right angle because we started with all right angles. We get the same area of the spiral:

$$\frac{n-2}{n}\pi - \frac{\pi}{2} = \frac{\pi}{2} - \frac{2\pi}{n}.$$

From this example, we see that we can fix the starter angle at size  $\alpha$  and the area of the spiral becomes  $\pi - \alpha - \frac{2\pi}{n}$  because the telescopic series will still happen. If we use non-midpoints for the nested polygon vertices, we lose the nice isosceles structure. Yet, we keep the telescoping series! To see this in Figure 8, think of the two  $\beta$ 's as  $\beta_1$  and  $\beta_2$ . The calculation for the area of triangle  $BCD$ , from (4), becomes

$$\pi - (\gamma + \beta_1 + \beta_2) = -\gamma + (\pi - \beta_1 - \beta_2) = \delta - \gamma.$$

The limit of the series is  $\frac{n-2}{n}\pi - \alpha = \pi - \alpha - \frac{2\pi}{n}$ , the same answer as using midpoints. This makes sense because, even though we chose vertices off the midpoints, we still generate a spiral which stands for  $\frac{1}{n}$  of the area inside the first polygon.

## 5 Sum of side lengths

For hyperbolic triangle  $ABC$ , the hyperbolic trigonometric formulas (1) and (2) work nicely with our nested polygons. We can show that the hyperbolic length of one side of the first level polygon is bounded as  $n$  increases.

**Lemma 6** *For the  $n$ -sided, all right angled, regular, hyperbolic polygon,  $\lim_{n \rightarrow \infty} S_1 = 3$ .*

**Proof.** *From formula (2) applied to triangle  $OCE$ ,  $\sin \frac{\pi}{4} \sin \frac{\pi}{4} \cosh L_1 = \cos \frac{2\pi}{n} + \cos \frac{\pi}{4} \cos \frac{\pi}{4}$ . Solving for  $\cosh L_1 = S_1$  and taking the limit gives the desired result. ■*

To translate into our levels, we use  $A_k$  for the angle  $C$ . We will use  $L_k$  for the hyperbolic length of one side at level  $k$ . More often, we will use  $S_k = \cosh L_k$ . The isosceles structure of the triangle makes  $a = b = .5 \cosh^{-1}(S_k)$ . The side opposite angle  $A_k$  already has a name:  $S_{k+1}$ . (The reason  $\cosh^{-1} S_k$  turns up so much is that  $S_k$  is not hyperbolic length. We have to take  $\cosh^{-1} S_k$  to get hyperbolic length.) Substitution into formula (1) gives us our recursive formula for the hyperbolic length of a side of one of our nested hyperbolic polygons:

$$S_{k+1} = [\cosh^2(.5 \cosh^{-1}(S_k)) - (\cos A_k \sinh^2(.5 \cosh^{-1}(S_k)))].$$

Simplifications for the compositions  $\cosh^2(.5 \cosh^{-1}(S_k)) = \frac{S_k + 1}{2}$  and  $\sinh^2(.5 \cosh^{-1}(S_k)) = \frac{S_k - 1}{2}$  have been known since at least 1707 [Kennelly]. (The substitutions  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$  and  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$  from any calculus text provide the key step.) The simplifications lead to a shorter recursive formula:

$$S_{k+1} = \frac{1}{2}[S_k(1 - \cos A_k) + (1 + \cos A_k)]. \quad (5)$$

We are now ready to consider the sum of one side from each level.

**Theorem 1**  $\sum_{k=1}^{\infty} \cosh^{-1}(S_k) < \infty$  for fixed  $n$ , the number of sides in the polygon.

**Proof.** Starting with the ratio test, we apply our recursive formula (5) to the numerator. Since all the subscripts have become  $k$ , we can suppress the subscripts. We are going to treat both  $S$  and  $A$  as functions of  $k$ . So, for example,  $S' = \frac{dS}{dk}$ .

$$\lim_{k \rightarrow \infty} \frac{\cosh^{-1}(S_{k+1})}{\cosh^{-1}(S_k)} = \lim_{k \rightarrow \infty} \frac{\cosh^{-1}(\frac{1}{2}[S(1 - \cos A) + (1 + \cos A)])}{\cosh^{-1}(S)}.$$

Since the limiting case is the center point  $O$ , we know the numerator and denominator go to 0. So we apply L'Hopital's rule with  $k$  as the independent variable.

$$\lim_{k \rightarrow \infty} \frac{\cosh^{-1}(\frac{1}{2}[S(1 - \cos A) + (1 + \cos A)])}{\cosh^{-1}(S)}$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \frac{\frac{1}{2} [S'(1 - \cos A) + (S - 1)A' \sin A] \sqrt{S^2 - 1}}{\sqrt{\left[\frac{S(1 - \cos A) + (1 + \cos A)}{2}\right]^2 - 1}} \frac{1}{S'} \\
&= \lim_{k \rightarrow \infty} \frac{[S'(1 - \cos A) + (S - 1)A' \sin A] \sqrt{S^2 - 1}}{\sqrt{[S^2(1 - 2 \cos A + \cos^2 A) + 2S(1 - \cos^2 A) + (1 + 2 \cos A + \cos^2 A)] - 4}} \frac{1}{S'} \\
&= \lim_{k \rightarrow \infty} \frac{[S'(1 - \cos A) + (S - 1)A' \sin A] \sqrt{S^2 - 1}}{\sqrt{[\cos^2 A(S - 1)^2 - 2 \cos A(S^2 - 1) + (S + 3)(S - 1)]}} \frac{1}{S'} \\
&= \lim_{k \rightarrow \infty} \frac{[S'(1 - \cos A) + (S - 1)A' \sin A] \sqrt{S + 1}}{\sqrt{[\cos^2 A(S - 1) - 2 \cos A(S + 1) + (S + 3)]}} \frac{1}{S'}.
\end{aligned}$$

The  $\lim_{k \rightarrow \infty} S = 1$ ; we see the factors of  $\sqrt{S - 1}$  have canceled by division in the last step, so we can get ready to take the limit. Before we can let the  $(S - 1)A' \sin A$  go to zero, however, we need to check that  $\lim_{k \rightarrow \infty} \frac{A' \sin A}{S'} < \infty$ . This is worth a look because the number  $S_k$  approaches 1; so  $S'$  can get small quickly.

We will now make sure that the factor  $A'$  gets small quickly enough to handle it. We can rewrite our hyperbolic trigonometry formula (3) for  $A = 2(\sin^{-1}(\frac{\cos(\frac{\pi}{n})}{\sqrt{\frac{S+1}{2}}}))$  (subscripts

suppressed). Then we find  $\frac{dA}{dS} = \frac{-\sqrt{2} \cos(\frac{\pi}{n})}{(S + 1) \sqrt{S + 1 - 2 \cos^2(\frac{\pi}{n})}}$  by the Chain Rule. The  $\lim_{k \rightarrow \infty} S = 1$  implies that this fraction is finite for any fixed  $n$ . We can finish taking our limit.

So the difficult-looking factor  $A' \sin A$  cancels in the limit and we find  $\lim_{k \rightarrow \infty} \frac{\cosh^{-1}(S_{k+1})}{\cosh^{-1}(S_k)} =$

$$\lim_{k \rightarrow \infty} \frac{\sqrt{1 - \cos A}}{\sqrt{2}} < 1 \text{ for any fixed } n. \quad \blacksquare$$

## 6 The constructible, regular, all right angled hyperbolic polygons

Our constructions give us one more result. Recall that our construction method in Section 3 begins with an angle. If that angle is constructible, then the resulting regular hyperbolic polygon is constructible. The constructible angles correspond to the constructible regular Euclidean polygons, which are known. Specifically, an  $n$ -sided regular Euclidean polygon is constructible if and only if  $n$  is a product of distinct Fermat primes and a power of 2. A Fermat prime is a prime number of the form  $2^m + 1$ . Many sources give a detailed investigation of the rich history of this result, [Venema] among them.

**Theorem 2** *The constructible regular hyperbolic polygons with all right angles have the same number of sides as the constructible Euclidean regular polygons.*

**Proof.** *We can show that the family of constructible regular hyperbolic polygons is the same as the family of constructible Euclidean regular polygons. In this case, by "family," we*

mean the number of sides, more than four. If we can construct a regular  $n$ -sided Euclidean polygon, we can construct a regular, right angled hyperbolic polygon from it. We gave its construction in Section 3.

In order to show the families are the same, we suppose we could construct a regular,  $n$ -sided, right angled hyperbolic polygon where  $n$  is not a power of 2 times a product of distinct Fermat primes, like  $n = 7$ . In other words, suppose we have a construction for a regular, right angled, hyperbolic polygon whose Euclidean version is not in the constructible family. If this construction has  $O$  as its center, we can draw Euclidean segments connecting consecutive centers of circles containing hyperbolic sides to construct a regular Euclidean polygon, which has been proved impossible.

If this hyperbolic construction is off-center, the hyperbolic sides will not be congruent Euclidean arcs or segments. The centers of the circles containing these Euclidean arcs will not be spaced symmetrically around  $O$ . Steps exist, however, which will center the drawing, using compass and straightedge [Goodman-Strauss]. We can translate the off-center polygon to the location we use in this paper. Then we are right back to the previous paragraph, constructing an impossible Euclidean polygon. Therefore, our assumption that we can construct a regular, right angled hyperbolic polygon from outside the Euclidean family leads to a contradiction.

■

Right angled hyperbolic polygons appear in the study of Riemann surfaces, as mentioned in the paper by Castro and Martinez [Castro/Martinez] These authors also mention that only a few papers exist which study these objects. The Euclidean versions of this problem have received more attention, in [Emmendorfer et al] and [Stewart].

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